HOMOLOGICAL PROPERTIES OF GENERIC MATRIX RINGS

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ABSTRACT

It is shown that the ring of two 2×2 generic matrices over a field has infinite global dimension. It is also proved that there is a non-free projective module over that ring. Finally, the authors show that the trace ring of that generic matrix ring is an iterated Ore extension from which it follows that the trace ring has global dimension five and that the finitely-generated projective modules are stably free.

Let k be an infinite field and $k\{x_1,\ldots,x_n\}$ the free algebra in n variables over k. If I_m is the ideal of all identities of $m\times m$ matrices in n variables then $k\{x_1,\ldots,x_n\}/I_m$ is called the ring of n generic $m\times m$ matrices. This ring is a domain of PI degree m. An often more convenient description of this ring is obtained by considering the polynomial ring $A=k[t^l_{ij}: 1\leqq i,j\leqq m,\ 1\leqq l\leqq n]$ and the matrices $X_l=(t^l_{ij})\in M_m(A)$. Then $k\{x_1,\ldots,x_n\}/I_m$ is just the k-algebra generated by the $\{X_l\}$. For these remarks and other background, the reader is referred to [9].

The generic matrix ring occupies a position of importance within PI theory similar to that of the free algebra and the commutative polynomial ring within their respective areas. However little is known about its homological properties. In this paper we consider the ring $R=k_{(2)}\{a,b\}$ of two generic 2×2 matrices and prove, among other things, that *R has infinite global dimension.* This answers a question of Procesi [7]. We also produce a *finitely generated, non-free projective R-module P that satisfies $P\oplus R\cong R\oplus R$.*

A useful tool for studying a prime PI ring is its trace ring (this is obtained by adjoining to the ring the coefficients of the reduced characteristic polynomial of

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its elements, considered as elements in the quotient ring of the ring). Let T be the trace ring of $R = k_{(2)}\{a, b\}$. One of our main observations is that *T is an iterated Ore extension*, from which it follows easily that *T has global dimension 5, is a maximal order, and all its finitely generated projective modules are stably free.* However, like R, T does possess non-free projective modules.

In this paper we will not need much of the general theory of PI rings, but rather we will use the concrete description of R and T available in [3].

1. Preliminaries

Our basic reference is [3]. Fix an infinite field k. Then throughout this note we use the following notation: $R = k_{(2)}\{a, b\}$ is the ring of two generic 2×2 matrices, with trace ring T. Let tr and det stand for (reduced) trace and determinant, respectively. Write $z = ab - ba$, $s = \operatorname{tr}(a)$, $t = \operatorname{tr}(b)$, $d = \det(a)$, $e = \det(b)$ and $f = \operatorname{tr}(ab)$, and observe that these last five elements do indeed belong to T.

The following facts from [3] will prove useful, and will usually be used without further reference.

(1.1) The centre, $Z(T)$, of T is the polynomial ring in the five variables, s, t, d, e and f.

(1.2) T is a free $Z(T)$-module, with basis $1, a, b, ab$.

(1.3) $d = -a^2 + sa$ and $e = -b^2 + tb$.

(1.4) $z^2 = f^2 - stf + s^2e + t^2d - 4de \in Z(T)$.

(1.5) $za = -az + sz$ and $zb = -bz + tz$.

(1.6) $f = ba + ab + st - ta - sb$.

2. Ore extensions and the trace ring

Let A be a ring, σ an automorphism of A and δ a σ-derivation (that is, $\delta(rs) = r\delta(s) + \delta(r)s^\sigma$ for all $r, s \in A$). Then the *Ore extension* $A[x; \sigma, \delta]$ of A is the ring which, additively, is isomorphic to the polynomial ring $A[x]$ but with multiplication defined by $rx = xr^\sigma + \delta(r)$ for $r \in A$. The aim of this section is to show that T can be realized as an iterated Ore extension. All our results concerning T will then follow from this result. We will use the following easy criterion for a ring to be an Ore extension.

LEMMA 2.1. [1, Theorem 1, p. 438] *Let $A \subset B$ be rings and $x \in B$ be such that $B = A\langle x\rangle$. Suppose that there exist set-theoretic homomorphisms σ and δ of A, such that:*

(i) *σ is a bijection,*

(ii) *for all $r \in A$, $rx = xr^\sigma + \delta(r)$,*

(iii) *every element* $b \in B$ *can be uniquely written as* $b = \sum_1^m x^i a_i$ *for some* $a_i \in A$ *and integer n.*

Then σ *is an automorphism and* δ *a* σ*-derivation of* A *such that* $B \cong A[x;\sigma,\delta]$.

THEOREM 2.2. *T is an iterated Ore extension.*

PROOF. By (1.1) and (1.4), $R_1 = k\langle z, s, t\rangle$ is a commutative polynomial ring and so is certainly an iterated Ore extension. Set $R_2 = R_1\langle a\rangle$. Let $r \in R_1$ and write $r = \sum z^i f_i$ for some $f_i \in k[s,t]$. Then, by (1.5), $ra = ar^\sigma + \delta(r)$ where σ, δ are maps such that $r^\sigma = \sum(-1)^i z^i f_i$. Thus, only the uniqueness in condition (iii) of Lemma 2.1 is not obvious. Notive that $k[s,t,z^2]^*$ is a central Ore set in R_1 and the resulting localization is just $V = k(s,t,z)\langle a\rangle$. Now suppose that $\sum a^i r_i = 0$ for some $r_i \in R_1$, not all of which are zero. Then V is a finite $k(s,t,z)$-module. But, by (1.3), $d \in R_2$ and so V contains the polynomial extension $k(s,t,z)[d]$; which is absurd. Thus Lemma 2.1 can indeed be applied to prove that R_2 is an Ore extension of R_1.

The observations of Section 1 show that $T = R_2\langle b\rangle$. The proof of the last paragraph can be easily modified to prove that T is an Ore extension, $T = R_2[b;\tau,\varepsilon]$ of R_2 and complete the proof.

A number of properties of T are immediate consequences of Theorem 2.2, as we now illustrate. The *global dimension* of a ring A will be written gl dim A, and the *homological dimension* of an A-module M will be denoted hd M.

COROLLARY 2.3. (i) gl dim $T = 5$.

(ii) *T is a maximal order.*

(iii) *T is stably free; that is, if P is a finitely generated projective T-module, then* $T \oplus R^{(n)} \cong R^{(m)}$ *for some integers n and m.*

PROOF. (i) T is a free module over $Z(T)$, which is a polynomial ring in 5 variables. Thus, by [4], gl dim $T \geqq 5$. The other direction follows from the theorem combined with [2, Remark (iii)].

(ii) Use [6, Proposition 2.5, p. 93].

(iii) This follows from the results on the K-theory of graded rings which are well-known but inaccessible. See [8] and the forthcoming [5].

Since K dim $T = 5$, Corollary 2.3(iii) combined with [10] proves the following. If P is a finitely generated, projective T-module of uniform dimension at least 5, then P is free. This raises the obvious question of whether all finitely generated projective T-modules are free.

COROLLARY 2.4. *Let $A = R$, the ring of two generic 2×2 matrices, or $A = T$, its trace ring. Then there exists a non-free, projective, right ideal P of A satisfying $P \oplus A \cong A \oplus A$.*

PROOF. Let $r = 1 + zt$ and $J = bA + rA$. Then $zt^2 = rb + b(zt-1) \in J$. Since $1 = r(1-zt) + (zt^2)z$, this implies that $J = A$. Thus, from the short exact sequence

$$0 \to bA \cap rA \to bA \oplus rA \to bA + rA \to 0,$$

we see that $P = bA \cap rA$ is a projective A-module satisfying $P \oplus A \cong A \oplus A$. Thus to prove the corollary we need only show that P is not cyclic.

Let $\mathscr{C} = \{z^{2n}\}$; so $\mathscr{C}$ is a central Ore set in both T and R. Furthermore, the resulting localization

$$A_{\mathscr{C}} = R_{\mathscr{C}} = T_{\mathscr{C}} = (R_2)_{\mathscr{C}}[b\,;\tau,\varepsilon]$$

is still an Ore extension (here R_2 is the ring constructed in the proof of Theorem 2.2). Since $P_{\mathscr{C}} = bA_{\mathscr{C}} \cap rA_{\mathscr{C}}$, to prove that P is not cyclic, it suffices to prove that $P_{\mathscr{C}}$ is not cyclic. But this is now a special case of [11, Theorem 1.2].

We remark that R is also a stably free ring. This will appear in a forthcoming paper of Severino Collier Coutinho.

3. Global dimension of generic matrices

In this final section we consider the global dimension of $R = k_{(2)}\{a, b\}$. We begin with some elementary lemmas. Observe that zT is an ideal of T. Throughout this section, $^-$ will denote the image of elements of T (or R) in $\bar{T} = T/zT$. By the comments of Section 1, $\bar{T} = k[\bar{a}, \bar{b}, \bar{s}, \bar{t}]$ is a commutative polynomial ring in the four named indeterminates.

LEMMA 3.1. (i) $RzR = zT \cap R = zT = \Sigma\{zs^it^jR : i, j \geqq 0\}$.

(ii) $tT \cap R = tRzR$.

(iii) $\bar{t}\bar{T} \cap \bar{R} = 0$.

PROOF. (i) is just [3, Lemma 2]. The other two parts follow from this and the fact that $\bar{T}$ is the polynomial ring $\bar{T} = \bar{R}[\bar{s}, \bar{t}]$.

LEMMA 3.2. *There exists a short exact sequence*

$$(*) \qquad 0 \longrightarrow RzR \xrightarrow{\ \theta\ } R \oplus R \xrightarrow{\ \phi\ } zR + ztR \longrightarrow 0.$$

PROOF. ϕ is the map defined by $(f, g) \to zf + zfg$ and so $\operatorname{Ker} \phi \cong \{g \in R : ztg \in zR\}$. Now, as $t \in Z(T)$, $ztRzR = zRztR \subset zR$. Thus $\operatorname{Ker} \phi \supseteq RzR$. The inverse inclusion is just Lemma 3.1(ii).

LEMMA 3.3. *$RzR/(zR + ztR) \cong \bigoplus^{\infty} R/RzR$, as a right R-module.*

PROOF. Suppose first that $\Sigma\, zs^it^jr_{ij} = 0$ for some $r_{ij} \in R$. Then $\Sigma\, s^it^jr_{ij} = 0$ and $\Sigma\, \bar{s}^i\bar{t}^j\bar{r}_{ij} = 0$ in $\bar{T} = T/zT$. But $\bar{T} = k[\bar{a}, \bar{b}, \bar{s}, \bar{t}]$ and each $\bar{r}_{ij} \in \bar{R} = k[\bar{a}\bar{b}]$. This forces each $\bar{r}_{ij} = 0$. That is, by Lemma 3.1(i) we have $r_{ij} \in zT \cap R = RzR$ for each i and j.

Now return to the proof of the lemma. By Lemma 3.1(i) $(RzR)^2 \subseteq zR$ and so we may consider RzR/zR as a right R/RzR-module. Since $RzR = \Sigma\, zs^it^jR$, the comments of the last paragraph show that RzR/zR is a free R/RzR-module with basis $\{zs^it^j : i, j \geqq 0$ with i, j not both zero$\}$. This clearly suffices to prove the lemma.

We now have all the pieces for:

THEOREM 3.4. $\operatorname{gl\,dim} R = \infty$.

PROOF. We will prove that $\operatorname{hd}_R R/RzR = \infty$. Let $\operatorname{hd}(zR + tzR) = n$. Note that the short exact sequence (*) cannot be split, since RzR is not finitely generated. Thus $n \geqq 1$. So, by (*) again, $\operatorname{hd}(RzR) = n - 1$ and $\operatorname{hd} R/RzR = n$. But Lemma 3.3 provides a short exact sequence

$$0 \to zR + tzR \to RzR \to \bigoplus R/RzR \to 0$$

in which the terms have homological dimension n, $(n - 1)$ and n respectively. This is only possible if $n = \infty$.

There is very little known about the homological properties of generic matrices, beyond what has been said here. Presumably all rings of generic matrices have infinite global dimension, although it is possible that the finitistic dimension remains finite.

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